



Aggregation and residuation

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AGGREGATION and RESIDUATION

Bruno LECLERC¹ and Bernard MONJARDET²

Abstract

In this note we give a characterization of meet-projections in simple atomistic lattices that generalizes results on the aggregation of partitions in cluster analysis.

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1 Introduction

In his celebrated 1951 book (*Social Choice and Individual Values*) Arrow proved that a rule to aggregate individual preferences into a collective preference, and satisfying some apparently natural conditions can be "dictatorial". When the n individual preferences are modelled by linear orders his result comes back to an "axiomatic" characterization of *projections* i.e., of a rule mapping always a n -tuple (L_1, \dots, L_n) of linear orders into a i -coordinate L_i . A crucial property to obtain Arrow's result is the so-called *independence axiom* saying that the collective preference on two alternatives must only depend on the individual preferences on these two alternatives. When applied to other types of relations like partial orders or equivalences this same independence property leads to characterizations of *meet-projections*: the collective relation is a meet of some individual relations (in social choice theory, such a rule is called "oligarchic").

The sets of partial orders or of equivalences are lattices (for the inclusion order³). We have shown (Monjardet 1990, Leclerc and Monjardet 1995) that the oligarchic results obtained by Brown (1975) on partial orders, by Mirkin and Leclerc (1975, 1984) or Neumann and Norton (1986) on equivalences (as well as other similar results) are applications of a general result on the aggregation of elements of a lattice (satisfying some properties). In this note, we show that one can obtain this general result by replacing the *decisivity* property (the latticial form of the independence property) by a purely latticial property, namely a *residuation* property. On the one hand this result generalizes results obtained in the case of partitions by Dimitrov, Marchant and Mishra (2009) and Chambers and Miller (2010). On the other hand it gives a characterization of meet-projections in simple atomistic lattices.

2 Preliminaries

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³ With a greatest element added to the semilattice of partial orders.

We recall some notations, definitions and results on lattice and latticial consensus theories.

Throughout this paper L denotes a finite lattice, $(L, \wedge, \vee, 0_L, 1_L)$. The set of all the *join-irreducible* elements (i.e. of the elements not join of elements different from themselves) of L is denoted by J .

A lattice is *atomistic* if all its join-irreducible elements are atoms (i.e., elements covering 0_L).

Let F be a map from a lattice L to a lattice L' (with order \leq' and operations \wedge' and \vee'):

F is a \wedge -*morphism* (respectively, a \vee -*morphism*) if for all x, y in L , $F(x \wedge y) = F(x) \wedge' F(y)$ (respectively $F(x \vee y) = F(x) \vee' F(y)$). Then such morphisms are *isotone* maps (i.e., $x \leq y$ implies $F(x) \leq' F(y)$).

F is a *residual map* (respectively, a *residuated map*) if F is a \wedge -morphism satisfying $F(1_L) = 1_{L'}$ (respectively a \vee -morphism satisfying $F(0_L) = 0_{L'}$).

It is well known that if F is a residual map from L to L' there exists a unique residuated map G from L' to L , such that *Pickert's relation* is satisfied: for all $x \in L, x' \in L'$,

$$x' \leq F(x) \Leftrightarrow G(x') \leq x,$$

and such that GF is *reductive* (i.e., $x \geq GF(x)$) and FG is *extensive* (i.e., $x \leq FG(x)$). Moreover the two images sets $GF(L)$ and $FG(L')$ are two isomorphic lattices (See Blyth and Janowitz 1972 or Caspard, Leclerc and Monjardet 2007 for details on residuation).

We come now to definitions and results on latticial consensus theory. In this theory the objects to be aggregated are the elements of a lattice L .

A *consensus* (or *aggregation*) *function* on L is a mapping F from L^n to L : it associates an element $x = F(\Pi)$ of L with each n -tuple $\Pi = (x_1, \dots, x_n)$ of elements of L (so, it is a n -ary operation on L).

In particular, a consensus function F from L^n to L is a *meet-projection* if there exists $\emptyset \subseteq M \subseteq N$ such that for every $\Pi \in L^n$, $F(\Pi) = \bigwedge_{i \in M} x_i$. Observe that if $M = \emptyset$, F is the constant function F^1 mapping each n -tuple $\Pi \in L^n$ into the greatest element 1_L of L .

For $\Pi = (x_1, \dots, x_n) \in L^n$ and $x \in L$, we write $N_x(\Pi) = \{i \in N : x \leq x_i\}$. In particular, for a consensus function F , we define several properties based on the sets $N_j(\Pi), j \in J$.

A consensus function F on L is *decisive* (D) if for every $j \in J$ and for all $\Pi, \Pi' \in L^n$,

$$[N_j(\Pi) = N_j(\Pi')] \Rightarrow [j \leq F(\Pi) \Leftrightarrow j \leq F(\Pi')]$$

A consensus function F on L is *neutral monotonic* (NM) if for all $j, j' \in J$ and for all $\Pi, \Pi' \in L^n$,

$$[N_j(\Pi) \subseteq N_{j'}(\Pi')] \Rightarrow [j \leq F(\Pi) \Rightarrow j' \leq F(\Pi')]$$

Observe that this property implies the so-called *monotonicity* (when $j = j'$) and *neutrality* (when $[N_j(\Pi) = N_{j'}(\Pi')]$) properties, as well as the decisivity property.

The following (easy to prove) result will be useful in the sequel.

Lemma

Let F be a neutral monotonic consensus function on $L, j \in J, x \in L$ and $\Pi, \Pi' \in L^n$ such that $j \leq$

$F(\Pi)$ and $N_x(\Pi') \supseteq N_j(\Pi)$. Then, $x \leq F(\Pi')$.

A consensus function F is *Paretian* (P) if for every $\Pi \in L^n$,

$$N_j(\Pi) = N \Rightarrow j \leq F(\Pi)$$

It is clear that such axioms are abstract forms of "Arrowian" properties. For example, decisivity corresponds to independence.

We will also use classical ordinal or algebraic axioms. Obviously L^n is a lattice with $\Pi \wedge \Pi' = (x_1 \wedge x'_1, \dots, x_n \wedge x'_n)$, $\Pi \vee \Pi' = (x_1 \vee x'_1, \dots, x_n \vee x'_n)$.

So F is a \wedge -morphism (respectively, a \vee -morphism) if for all $\Pi, \Pi' \in L^n$, $F(\Pi \wedge \Pi') = F(\Pi) \wedge F(\Pi')$ (respectively, $F(\Pi \vee \Pi') = F(\Pi) \vee F(\Pi')$).

Let us denote by x^* the constant n -tuple (x, \dots, x) . Then the greatest (respectively, least) element of the lattice L^n is 1^* (respectively, 0^*), and F is a residual map (respectively, a residuated map) if F is a \wedge -morphism satisfying $F(1_L^*) = 1_L$ (respectively, a \vee -morphism satisfying $F(0_L^*) = 0_L$).

We say that F is *meet-compatible* if for every $\Pi = (x_1, \dots, x_n) \in L^n$

$$\bigwedge \{x_i, i \in N\} \leq F(\Pi).$$

One easily checks that the Paretian and the meet-compatibility properties are equivalent.

The results obtained in latticial consensus theory depend on the structural properties of the involved lattices and, especially, on the properties of a *dependence relation* δ defined on the set J of the join-irreducible elements of L . For j and j' in J we write:

$$j \delta j' \text{ if } j \neq j' \text{ and there exists } x \in L \text{ such that } j, j' \not\leq x \text{ and } j < j' \vee x$$

Observe that this relation δ contains the strict order relation between the join-irreducible elements (if $j < j'$, then $j < j' \vee 0_L$). One easily shows that δ equals this order relation if and only if L is a *distributive lattice* (for other properties of δ see Caspard and Monjardet 1997).

The relation δ defines an oriented graph on the set J of all the join-irreducible elements of L . The lattice L is said δ -strong if this graph is *strongly connected* (i.e., if for any ordered pair (j, j') of join-irreducible elements, there exists a path from j to j' in this graph.)

3 The results

In the proof of the following theorem, we adopt special notations for some n -tuples that will occur frequently. Let, for instance, (A, B, C) be a partition of the set N . $\Pi = (A: x, B: y, C: z)$ is the n -tuple for which for every i in A (respectively, in B, C) $x_i = x$ (respectively, y, z)

Theorem Let L be a δ -strong atomistic finite lattice and $F: L^n \rightarrow L$ a consensus function. The following are equivalent:

- (1) *F is decisive and Paretian;*

- (2) F is neutral monotonic and it is not equal to F^0 ;
- (3) F is a \wedge -morphism and meet-compatible;
- (4) F is a residual map and $F(j^*) \geq j$ for any $j \in J$;
- (5) F is a meet projection.

Proof

(1) \Leftrightarrow (2) This is proved for any δ -strong finite lattice in Monjardet (1990) (see also Leclerc and Monjardet 1995).

(2) \Rightarrow (3)

By the above equivalence F is Paretian, and so meet-compatible (since it has been above observed that these two properties are equivalent). Assume that F is not a \wedge -morphism i.e., that there exists $\Pi, \Pi' \in L^n$ such that $F(\Pi \wedge \Pi') \neq F(\Pi) \wedge F(\Pi')$ and, equivalently, $\{j \in J : j \leq F(\Pi \wedge \Pi')\} \neq \{j \in J : j \leq F(\Pi) \wedge F(\Pi')\}$.

First case: there exists an atom $j \in J$ such that $j \leq F(\Pi \wedge \Pi')$ and $j \not\leq F(\Pi) \wedge F(\Pi')$. So $j \not\leq F(\Pi)$ or $j \not\leq F(\Pi')$. Assume, for instance $j \not\leq F(\Pi)$ and consider $N_j(\Pi)$ and $N_j(\Pi \wedge \Pi')$. If $N_j(\Pi \wedge \Pi') = N_j(\Pi)$ decisivity would imply $j \leq F(\Pi)$, a contradiction. So, one has $N_j(\Pi \wedge \Pi') \subset N_j(\Pi)$. Since δ is strong, there exists $j' \in J$ with $j \delta j'$ i.e., such that there exists $x \in L$ with $j, j' \not\leq x$ and $j < j' \vee x$.

Consider then the (well defined) following n-tuple :

$$\Pi'' : [N_j(\Pi \wedge \Pi') : j' \vee x ; N_j(\Pi) \setminus N_j(\Pi \wedge \Pi') : j ; N \setminus N_j(\Pi) : 0_L].$$

Then $N_j(\Pi'') = N_j(\Pi)$ and $j \not\leq F(\Pi)$ implies (by decisivity) $j \not\leq F(\Pi'')$.

$N_j(\Pi'') = N_x(\Pi'') = N_j(\Pi \wedge \Pi')$ imply by neutral monotony (owing the Lemma higher up) $j' \vee x \leq F(\Pi'')$

Then, $j < j' \vee x \leq F(\Pi'')$, a contradiction.

Second case: there exists an atom $j \in J : j \not\leq F(\Pi \wedge \Pi')$ and $j \leq F(\Pi) \wedge F(\Pi')$.

So, $j \leq F(\Pi), j \leq F(\Pi')$ and (by the Paretian property) there exists $i \in N$ such that $j \not\leq x_i \wedge x'_i$.

$N_j(\Pi \wedge \Pi') = N_j(\Pi) \cap N_j(\Pi') \subset N_j(\Pi)$ and $\subset N_j(\Pi')$ (since, if for example, $N_j(\Pi \wedge \Pi') = N_j(\Pi)$ decisivity implies $j \leq F(\Pi \wedge \Pi')$ a contradiction).

Let $j' \in J$ such that $j' \delta j$ i.e., such that there exists $x \in J$ with $j, j' \not\leq x$ and $j' < j \vee x$.

Consider then the (well defined) following n-tuple Π'' :

$$[N_j(\Pi) \setminus N_j(\Pi \wedge \Pi') : j ; N_j(\Pi') \setminus N_j(\Pi \wedge \Pi') : x ; N_j(\Pi \wedge \Pi') : j \vee x ; N \setminus (N_j(\Pi) \cup N_j(\Pi')) : 0_L].$$

$N_j(\Pi'') = N_j(\Pi)$ and $j \leq F(\Pi)$ imply (by decisivity) $j \leq F(\Pi'')$.

$N_x(\Pi'') = N_j(\Pi')$ and $j \leq F(\Pi')$ imply by neutral monotony $x \leq F(\Pi'')$.

Then $j' < j \vee x \leq F(\Pi'')$ and $N_j(\Pi \wedge \Pi') = N_j(\Pi'')$ implies by neutrality $j \leq F(\Pi \wedge \Pi')$, a contradiction.

(3) \Rightarrow (4) F is residual since F is a \wedge -morphism satisfying $F(1_L^*) = 1$ (by meet-compatibility). And

F meet-compatible implies $j \leq F(j^*)$.

(4) \Rightarrow (5) Consider an atom $j \in J$ and the residuated map G of the residual map F . Since $j \leq F(j^*)$, the isotony of G and the reductivity of GF imply $G(j) \leq GF(j^*) \leq j^*$. So for any $i \in N$, $G_i(j) \in \{0, j\}$, where $G_i(j)$ is the i -th component of $G(j)$. Write $M(j) = \{i \in N: G_i(j) = j\}$.

Let $j, j_1, \dots, j_r \in J$ such that $j \leq \bigvee_{1 \leq k \leq r} j_k$ and the set $\{j_1, \dots, j_r\}$ is minimal with that inequality. Then by isotony and join preservation of G , one has $G(j) \leq G(\bigvee_{1 \leq k \leq r} j_k) = \bigvee_{1 \leq k \leq r} G(j_k)$. So $G_i(j) = j$ implies $G_i(j_k) = j_k$ for all $j = 1, \dots, r$ and $M(j) \subseteq M(j_k)$.

Now consider j and j' in J such that $j \delta j'$ holds. Since every element x of L is a join of atoms, we can apply the previous considerations to obtain $M(j) \subseteq M(j')$. Since L is δ -strong, we get $M(j) = M(j') = M$, no matter of the considered pair j, j' .

The characterizations of the mappings G and F follow:

for any $x \in L$, since x is a join of atoms and G is join preserving, $G_i(x) = x$ if $i \in M$ and $G_i(x) = 0$ if not,

for a n -tuple $\Pi = (x_1, \dots, x_n)$, if M is nonempty, one gets from the Pickert relation, $x \leq F(\Pi) \Leftrightarrow G(x) \leq \Pi \Leftrightarrow [\text{for any } i \in M, x \leq x_i] \Leftrightarrow x \leq \bigwedge_{i \in M} x_i$. If M is the empty set, F is the residual mapping associated to the corresponding G , that is F maps any n -tuple onto the greatest element 1_L of L (the meet for the empty set of indices). In both cases, the formula $F(\Pi) = \bigwedge_{i \in M} x_i$ holds.

(5) \Rightarrow (1) Obvious. □

Remarks

1 One observes that the condition $F(j^*) \geq j$ in (4) is a weakening of the meet-compatibility i.e., of the Paretian property.

2 Since F is a residual map and G the associated residuated map, the two lattices $GF(L^n)$ and $FG(L)$ are isomorphic. One easily checks that (if M is nonempty) $GF(L^n) = \{(x_1, \dots, x_n) : x_i = x \text{ (respectively, } 0_L) \text{ for every } i \in M \text{ (respectively, for every } i \notin M) \text{ and } FG(L) = L$.

Recall that a lattice L is called *simple* if its only congruences are the trivial one and L^2 .

Proposition

An atomistic lattice is strong if and only if it is simple.

Proof

This result comes immediately from the following two facts concerning the relation C defined on the set J of join-irreducibles of a lattice L by Day (1979). Following Freese and al (1995) we call

this relation a dependence relation and we denote it by D . In the definition below of D , j^r is the element covered by the join-irreducible j :

$$j D j' \text{ if } j \neq j' \text{ and there exists } x \in L \text{ such that } j < j' \vee x \text{ and } j \not\leq j^r \vee x$$

The first fact is Day's result according to a lattice is simple if and only if the dependence relation D is strongly connected. The second fact is that a lattice is atomistic if and only if the two dependence relations δ and D are equal (Caspard and Monjardet, 1997). \square

The consequence of the above theorem and proposition is the following characterization of meet-projections in simple atomistic lattices.

Corollary

A n -ary operation F on a simple atomistic lattice is a meet-projection if and only if it is a residual map satisfying $F(j^) \geq j$ for any $j \in J$.*

Obviously, all the above results can be dualized for simple coatomistic lattices.

4 Conclusion

The lattice of partitions of a set is a simple geometric lattice, so an atomistic and coatomistic lattice. The application of the above results to this simple atomistic lattice gives again the results obtained by Dimitrov, Marchant and Mishra (2009) and Chambers and Miller (2010). The dual results on this lattice gives a characterization of *join-projections* as a residuated map to compare with the characterizations given in Neumann and Norton (1986) and Leclerc and Monjardet (1995). Clearly, the interest of the abstract "axiomatic" latticial approach to aggregation theory is to give results applicable to several different problems. For instance, the above theorem gives a characterization of meet-projections ("oligarchic" consensus functions) for partial orders. The abstract latticial approach has been also introduced for aggregation procedures based on distances by Barthélemy and Janowitz-(1991) and it has been developped by several authors. A review of these works can be found in Day and McMorris (2003).

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